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Symmetry groups and conserved quantities for the harmonic oscillator

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Abstract. The complete eight-parameter symmetry group of the one-dimensional harmonic oscillator is investigated using the fact that the system is describable by a variational principle. It is found that only a five-parameter subgroup leaves the action integral invariant, thus yielding five conserved quantities, only two of which are functionally independent. These two conserved quantities determine the solutions, and correspond to a two-parameter Abelian subgroup. We also show that if a conserved quantity corresponds to a symmetry group by Noether's theorem, then the group transforms any solution into another solution possessing the same value of the conserved quantity. In addition, we find how the Lagrangian transforms under symmetry groups which do not preserve the action integral, leading to certain alternative Lagrangians for the harmonic oscillator.

1. Introduction

The complete symmetry group of the one-dimensional harmonic oscillator has recently been discussed by several authors (Anderson and Davison 1974, Wulfman and Wybourne 1976) and has been shown to be the eight-parameter Lie group SL(3, R). This work was carried out using the Lie theory of extended groups (Lie 1891, 1922, Cohen 1931) to study the invariance of the equation of motion. In this paper we show that additional insight is obtainable by utilising the fact that the harmonic oscillator equation is derivable from a variational principle. In particular, since Noether's theorem (Noether 1918) states that there is a close connection between the conservation laws and the symmetry properties of a system describable by a Lagrangian, it seems natural to attempt to correlate the conserved quantities for the harmonic oscillator with the group SL(3, R) mentioned above. We will find that only a certain five-parameter subgroup of SL(3, R) leads to conserved quantities via Noether's theorem, and of the five resulting constants of the motion, only two are functionally independent. These two independent constants of the motion correspond to a two-parameter Abelian subgroup of the five-parameter group, and are sufficient to determine the harmonic oscillator solutions. It thus appears that knowledge of the Abelian subgroup suffices to completely describe the harmonic oscillator.

These results are related to the fact that Noether's theorem associates a conserved quantity with each one-parameter group that leaves the action integral invariant. Although the full eight-parameter symmetry group leaves the equation of motion invariant, only the five-parameter subgroup satisfies the more stringent condition of preserving the action integral.

The invariance of the action integral requires certain transformation properties of the Lagrangian itself. We investigate the way the Lagrangian transforms under those groups which preserve the equation of motion but not the action integral, and show that these considerations lead to various alternative forms for the harmonic oscillator Lagrangian. In addition, we will use the Lagrangian formalism to derive the full symmetry group for the harmonic oscillator.

Another aspect of our investigation concerns the transformation properties of harmonic oscillator solutions. It is known that if a differential equation is invariant under the action of a group, then the group transforms solutions into solutions. Wulfman and Wybourne (1976) have given specific examples of the manner in which harmonic oscillator solutions are transformed among themselves by the action of the symmetry group. We supplement their conclusions by showing that if a conserved quantity corresponds to a symmetry group by virtue of Noether's theorem, then the symmetry group transforms any solution into another solution possessing the same value of the conserved quantity. This provides a criterion for deciding whether solutions are transformable into one another by an element of the five-parameter subgroup.

2. Symmetries and conserved quantities

For the sake of completeness, and to establish notation, we give in this section a brief outline of the theory of symmetry groups and conserved quantities. Since we are interested here primarily in the one-dimensional harmonic oscillator, we shall limit ourselves to Lagrangians dependent only on one space coordinate and time.

Let a physical system be described by a Lagrangian $L(q, \dot{q}, t)$ and let the action integral be

$$A = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, \mathrm{d}t, \tag{1}$$

where the integration is along a curve $q = q(t)$. Consider a one-parameter Lie group of transformations from the (q, t) variables to the (Q, T) variables, whose infinitesimal generator is the operator

$$G(q, t) = \xi(q, t)\frac{\partial}{\partial t} + \eta(q, t)\frac{\partial}{\partial q}. \tag{2}$$

The finite transformations of the group may be given in the form (taking into account the effect on the time derivatives of q):

$$Q = \mathrm{e}^{\theta G(q,t)} q, \tag{3a}$$

$$T = \mathrm{e}^{\theta G(q,t)} t, \tag{3b}$$

$$\dot{Q} = \mathrm{e}^{\theta E(q,\dot{q},t)} \dot{q}, \tag{3c}$$

where θ is the group parameter, and the operator $E(q, \dot{q}, t)$ is defined by

$$E(q, \dot{q}, t) = G(q, t) + (\dot{\eta} - \dot{q}\dot{\xi})\frac{\partial}{\partial \dot{q}}, \tag{4}$$

with

$$\dot{\eta}=\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial q}\dot{q}, \tag{5a}$$

$$\dot{\xi}=\frac{\partial\xi}{\partial t}+\frac{\partial\xi}{\partial q}\dot{q}. \tag{5b}$$

The operator E is the generator of the first extended group (Lie 1891, 1922, Cohen 1931), which gives the change induced in a function of q, $\dot{q}$, t by the action of the original group on q and t. If the variables q, $\dot{q}$, t in (1) are replaced by their expressions in terms of Q, $\dot{Q}$, T, the action integral becomes

$$A=\int_{T_1}^{T_2}\tilde{L}(Q,\dot{Q},T,\theta)\,\mathrm{d}T, \tag{6}$$

where the new Lagrangian is given by

$$\tilde{L}(Q,\dot{Q},T,\theta)=(\mathrm{e}^{-\theta E(Q,\dot{Q},T)}L(Q,\dot{Q},T))\left(\frac{\partial t}{\partial T}+\frac{\partial t}{\partial Q}\dot{Q}\right). \tag{7}$$

If it turns out that

$$\tilde{L}(Q,\dot{Q},T,\theta)=L(Q,\dot{Q},T), \tag{8}$$

then the action integral is said to be invariant, and the group transforms solutions of the Euler equation into solutions. In fact, solutions are transformed into solutions even if the more general condition

$$\tilde{L}(Q,\dot{Q},T,\theta)=L(Q,\dot{Q},T)+\dot{F}(Q,T,\theta) \tag{9}$$

holds, since the Euler equation is unaffected by the addition of a total time derivative to the Lagrangian. We therefore take (9) as our general definition of the invariance of the action integral.

Noether's theorem, applied to our case, states that whenever the action integral is invariant under the group (3), the solutions to Euler's equation admit the conserved quantity

$$\Phi=(\xi\dot{q}-\eta)\frac{\partial L}{\partial\dot{q}}-\xi L+f, \tag{10}$$

where f is a function of q and t. The proof of this theorem is generally carried out using the calculus of variations (Gelfand and Fomin 1963); however, for our present purposes it is more convenient to assume the above form for Φ and verify explicitly that it is a conserved quantity. Thus, if Φ is differentiated totally with respect to t, we obtain the identity

$$\dot{\Phi}=(\xi\dot{q}-\eta)\mathscr{F}-E\{L\}-\dot{\xi}L+\dot{f}, \tag{11}$$

where

$$\mathscr{F}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}}\right)-\frac{\partial L}{\partial q} \tag{12}$$

and

$$E\{L\}=\xi\frac{\partial L}{\partial t}+\eta\frac{\partial L}{\partial q}+(\dot{\eta}-\dot{q}\dot{\xi})\frac{\partial L}{\partial\dot{q}}. \tag{13}$$

If the Euler equation is satisfied (i.e. $\mathscr{F}=0$) and if ξ and η can be chosen such that

$$E\{L\}=-\dot{\xi}L+\dot{f}, \tag{14}$$

then

$$\dot{\Phi}=0, \tag{15}$$

and Φ is a constant of the motion. Equation (14) can be regarded as a specification of the way the function $L(q, \dot{q}, t)$ must transform under the action of the group if Φ is to be a constant of the motion. The relation between the condition (14) and the invariance of the action integral is easily found. If equation (9) is expanded to first order in θ (using (7) for $\tilde{L}$) we obtain

$$L-\theta(E\{L\}+\dot{\xi}L)+\ldots=L+\theta\dot{f}+\ldots. \tag{16}$$

Equating coefficients of θ yields equation (14).

To conclude this section, we prove that

$$E\{\Phi\}=0. \tag{17}$$

The significance of this result has to do with the transformation properties of solutions under the action of the group. The operator E acting on $\Phi(q, \dot{q}, t)$ gives the change in Φ (to first order) resulting from the action of the symmetry group on the variables q, t. But the symmetry group transforms a solution into a solution, so that $E\{\Phi\}$ measures the change in the conserved quantity Φ due to the replacement of one solution by another. If (17) holds, we may then conclude that any two solutions related by the given symmetry group possess the same value of the associated conserved quantity.

The proof of (17) is carried out by an explicit application of the operator E to the conserved quantity Φ. Using (14) and the easily proven identity

$$E\left\{\frac{\partial L}{\partial \dot{q}}\right\}=\frac{\partial}{\partial \dot{q}}E\{L\}+(\xi_t-\eta_q+2\xi_q\dot{q})\frac{\partial L}{\partial \dot{q}},$$

we obtain

$$\begin{aligned}E\{\Phi\}=&\left[(\xi\dot{q}-\eta)\frac{\partial \dot{f}}{\partial \dot{q}}+\xi f_t+\eta f_q-\xi\dot{f}\right]\\&+[E\{\xi\dot{q}-\eta\}-\xi\dot{\xi}\dot{q}+\eta\dot{\xi}-(\xi\dot{q}-\eta)(\eta_q-\xi_t-2\xi_q\dot{q})]\frac{\partial L}{\partial \dot{q}}\\&+\left[\xi\dot{\xi}-\xi\xi_t-\eta\xi_q-(\xi\dot{q}-\eta)\frac{\partial \dot{\xi}}{\partial \dot{q}}\right]L.\end{aligned} \tag{18}$$

Expanding the total time derivatives, using (5), we find eventually that each of the quantities in rectangular brackets vanishes; thus (17) is proved.

3. Conserved quantities for the harmonic oscillator

If the Lagrangian of a system is known, then (14) may be regarded as the condition on $\xi(q, t)$ and $\eta(q, t)$ which ensures that (2) generates a symmetry group for the system, and that (10) is the associated constant of the motion. For the harmonic oscillator

$$L=\dot{q}^2-q^2, \tag{19}$$

so that (14) takes the form

$$(2\eta q+\xi_t q^2+f_t)=(2\eta_t-\xi_q q^2-f_q)\dot q+(2\eta_q-\xi_t)\dot q^2-(2\xi_q)\dot q^3. \tag{20}$$

Equating to zero the coefficients of powers of $\dot q$ leads to a system of four partial differential equations, which can easily be shown to have the solution

$$\eta(q,t)=(G\cos 2t-H\sin 2t)q+E\cos t+F\sin t, \tag{21}$$

$$\xi(q,t)=(G\sin 2t+H\cos 2t)+C, \tag{22}$$

$$f(q,t)=-2(G\sin 2t+H\cos 2t)q^2+(F\cos t-E\sin t)2q. \tag{23}$$

The quantities G, H, E, F, and C are arbitrary constants. We therefore have a five-parameter set of solutions, from which we can construct five linearly independent group generators, each of the form $\xi(\partial/\partial t)+\eta(\partial/\partial q)$:

$$G_1=(\sin 2t)\frac{\partial}{\partial t}+(q\cos 2t)\frac{\partial}{\partial q}, \tag{24a}$$

$$G_2=(\cos 2t)\frac{\partial}{\partial t}-(q\sin 2t)\frac{\partial}{\partial q}, \tag{24b}$$

$$G_3=(\cos t)\frac{\partial}{\partial q}, \tag{24c}$$

$$G_4=(\sin t)\frac{\partial}{\partial q}, \tag{24d}$$

$$G_5=\frac{\partial}{\partial t}. \tag{24e}$$

These operators generate a five-parameter Lie group, as can be deduced from the commutation relations:

$$[G_1,G_2]=-2G_5, \qquad [G_5,G_1]=2G_2, \qquad [G_2,G_5]=2G_1, \tag{25a}$$

$$[G_3,G_4]=0, \tag{25b}$$

$$[G_3,G_1]=[G_2,G_4]=[G_5,G_4]=G_3, \tag{25c}$$

$$[G_1,G_4]=[G_3,G_5]=[G_2,G_3]=G_4. \tag{25d}$$

To each of the one-parameter subgroups corresponds a constant of the motion given by (13); explicitly we have

$$C_1=(\dot q^2-q^2)\sin 2t-2q\dot q\cos 2t, \tag{26a}$$

$$C_2=(\dot q^2-q^2)\cos 2t+2q\dot q\sin 2t, \tag{26b}$$

$$C_3=-2\dot q\cos t-2q\sin t, \tag{26c}$$

$$C_4=-2\dot q\sin t+2q\cos t, \tag{26d}$$

$$C_5=\dot q^2+q^2. \tag{26e}$$

At most two of these conserved quantities are independent; if we choose C_3 and C_4 as the independent quantities, it is easy to show that

$$C_1 = \tfrac{1}{2}C_3C_4, \tag{27a}$$

$$C_2 = \tfrac{1}{4}(C_3^2 - C_4^2), \tag{27b}$$

$$C_5 = \tfrac{1}{4}(C_3^2 + C_4^2). \tag{27c}$$

Eliminating $\dot{q}$ between C_3 and C_4 yields

$$q = \tfrac{1}{2}C_4 \cos t - \tfrac{1}{2}C_3 \sin t, \tag{28}$$

which is the general solution for the harmonic oscillator. Thus, the one-dimensional harmonic oscillator is completely specified by the two-parameter Abelian symmetry group generated by G_3 and G_4.

We have previously shown that two solutions connected by a symmetry transformation possess the same value of the associated constant of the motion. We illustrate this for the transformation generated by G_4, for which the group operator is $e^{\theta(\sin t)\partial/\partial q}$.

Let

$$q - A \cos t - B \sin t = 0 \tag{29}$$

be the general solution of the harmonic oscillator; then

$$e^{\theta(\sin t)\partial/\partial q}(q - A \cos t - B \sin t) = q - A \cos t + (\theta - B)\sin t. \tag{30}$$

Thus the group transforms a given solution into another solution possessing the same value of A. But the relevant constant of the motion is C_4, and using (29) we may show that $C_4 = 2A$. It therefore follows that the value of the conserved quantity is maintained under the symmetry transformation.

4. The complete symmetry group

In the previous section we have seen that a search for harmonic oscillator symmetry transformations via Noether's theorem leads to a certain five-parameter Lie group. However, it is known (Anderson and Davison 1974, Wulfman and Wybourne 1976) that the complete symmetry group of the harmonic oscillator is an eight-parameter group. Thus there must exist, in addition to the original five-parameter group, three additional one-parameter groups which transform solutions into solutions, but for which the transformation properties of the Lagrangian are such that (9) and (14) no longer hold. The complete symmetry group was first derived by Anderson (Anderson and Davison 1974) using Lie's method of extended groups; here we will use the Lagrangian formalism for this purpose. We will also determine the transformation properties of the Lagrangian under the three additional groups, and show that the procedure leads to certain alternative Lagrangians for the harmonic oscillator.

Consider the symmetry group whose generator is (2), and for which the generator of the first extended group is (4). Since the group transforms solutions into solutions, the Euler expression for the new Lagrangian $\tilde{L}$ (see equation (7)) must be a multiple of the Euler expression for L; that is, we must have

$$\frac{d}{dT}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}}\right) - \frac{\partial \tilde{L}}{\partial Q} = M(Q, \dot{Q}, T, \theta)\left[\frac{d}{dT}\left(\frac{\partial L}{\partial \dot{Q}}\right) - \frac{\partial L}{\partial Q}\right]. \tag{31}$$

From the left-hand side of equation (16) we obtain, to first order in θ:

$$\tilde{L} = L - \theta L', \tag{32a}$$

where

$$L' = E\{L\} + \dot{\xi} L. \tag{32b}$$

Using (32a) in (31), and again retaining only first-order terms, we find

$$\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\partial L'}{\partial \dot{Q}}\right) - \frac{\partial L'}{\partial Q} = V(Q, \dot{Q}, T)\left[\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\partial L}{\partial \dot{Q}}\right) - \frac{\partial L}{\partial Q}\right]. \tag{33}$$

The expression (33), with L' given by (32b), will now be utilised to derive the full eight-parameter symmetry group of the harmonic oscillator. (For notational consistency with § 3, we now revert to the use of lower case symbols q, $\dot{q}$, t.)

For the harmonic oscillator we have

$$L = \dot{q}^2 - q^2, \tag{34}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 2(\ddot{q} + q), \tag{35}$$

$$E\{L\} = 2\dot{q}\dot{\eta} - 2q\eta - 2\dot{\xi}\dot{q}^2. \tag{36}$$

Using (32b) we obtain

$$L' = 2\dot{q}\dot{\eta} - 2q\eta - \dot{\xi}\dot{q}^2 - \dot{\xi}q^2. \tag{37}$$

After some calculation, it can be shown that, for L' given by (37):

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L'}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} \\
&\quad = (4\eta_q - 2\xi_t - 6\xi_q\dot{q})(\ddot{q} + q) + [2\eta_{tt} + 4q\xi_t + 2\eta - 2q\eta_q] \\
&\qquad + [4\eta_{tq} - 2\xi_{tt} + 6q\xi_q]\dot{q} + [2\eta_{qq} - 4\xi_{tq}]\dot{q}^2 + [-2\xi_{qq}]\dot{q}^3.
\end{aligned} \tag{38}$$

It is clear that in order for (33) to be satisfied, each of the four square brackets in (38) must vanish. This yields a system of four partial differential equations possessing the eight-parameter solution

$$\xi = (A \cos t + B \sin t)q + (G \sin 2t + H \cos 2t) + C, \tag{39}$$

$$\eta = (-A \sin t + B \cos t)q^2 + (G \cos 2t - H \sin 2t)q + Dq + (E \cos t + F \sin t). \tag{40}$$

From this solution we can construct eight linearly independent group generators, consisting of the previously defined operators G_1, G_2, G_3, G_4, G_5, plus the three additional operators

$$G_6 = q\frac{\partial}{\partial q}, \tag{41}$$

$$G_7 = (q \sin t)\frac{\partial}{\partial t} + (q^2 \cos t)\frac{\partial}{\partial q}, \tag{42}$$

$$G_8 = (q \cos t)\frac{\partial}{\partial t} - (q^2 \sin t)\frac{\partial}{\partial q}. \tag{43}$$

These eight operators generate an eight-parameter Lie group, as can be verified by considering the following commutation relations, in conjunction with those of (25):

$$[G_6, G_7] = G_7, \qquad [G_6, G_8] = G_8, \qquad [G_7, G_8] = 0, \tag{44a}$$

$$[G_6, G_1] = [G_6, G_2] = [G_6, G_5] = 0, \tag{44b}$$

$$[G_6, G_3] = -G_3, \qquad [G_6, G_4] = -G_4, \tag{44c}$$

$$\begin{aligned} &[G_7, G_1] = -G_7, \qquad [G_7, G_2] = -G_8, \\ &[G_7, G_3] = -\tfrac{1}{2}(G_1 + 3G_6), \\ &[G_7, G_4] = \tfrac{1}{2}(G_2 - G_5), \\ &[G_7, G_5] = -G_8, \end{aligned} \tag{44d}$$

$$\begin{aligned} &[G_8, G_1] = G_8, \qquad [G_8, G_2] = -G_7, \\ &[G_8, G_3] = -\tfrac{1}{2}(G_2 + G_5), \\ &[G_8, G_4] = \tfrac{1}{2}(-G_1 + 3G_6), \\ &[G_8, G_5] = G_7. \end{aligned} \tag{44e}$$

The operators G_1 to G_8 may be expressed as linear combinations of the operators $X_1, X_2, \ldots, X_8$ found by Wulfman and Wybourne (1976), who also showed that the X_i generate the global Lie group SL(3, R).

It is perhaps of interest here to briefly compare the two methods of deriving the full symmetry group; namely, the Lagrangian formalism, and the Lie method of extended groups. For the Lie method it is necessary to calculate the generators of both the first and second extended groups, whereas the Lagrangian approach requires only $E(q, \dot{q}, t)$, the generator of the first extended group. The reason for the distinction lies in the fact that the Lie method investigates directly the transformation properties of the Euler differential equation, which contains a second derivative. The Lagrangian approach, on the other hand, deals with the transformation properties of the Lagrangian, which contains only a first derivative. Both methods, of course, lead to the same set of partial differential equations having the solution (39), (40).

From (44a) we note that the operators G_6, G_7, and G_8 generate a subgroup of the full symmetry group; from our method of derivation it is clear that this subgroup contains all symmetry transformations which do not lead to conserved quantities via Noether's theorem. In particular, the transformation properties of the Lagrangian which are specified by (9) and (14) cannot hold for the elements of this subgroup. We now investigate how the Lagrangian does transform under this subgroup.

For the generator G_6 we have

$$\xi = 0, \tag{45a}$$

$$\eta = q, \tag{45b}$$

$$E = q\frac{\partial}{\partial q} + \dot{q}\frac{\partial}{\partial \dot{q}}, \tag{45c}$$

so that for the harmonic oscillator Lagrangian (19) we obtain, from (36):

$$E\{L\} = 2L. \tag{46}$$

From (37) we then have

$$L' = 2L,$$

and (32*a*) gives for the new Lagrangian, to first order:

$$\tilde{L} = (1 - 2\theta)L. \tag{47}$$

Equation (47) states that, to first order, the new Lagrangian is simply a multiple of the old. In fact, it is easy to show that this holds for all orders, and not just for the infinitesimal transformation. The finite transformation associated with G_6 may be written

$$Q = e^{\theta} q, \tag{48a}$$

$$T = t, \tag{48b}$$

$$\dot{Q} = e^{\theta} \dot{q}. \tag{48c}$$

Using this in (7), with $L = \dot{Q}^2 - Q^2$, we find

$$\tilde{L} = e^{-2\theta} L. \tag{49}$$

Expanding (49) to first order gives (47), as of course it should.

Comparing (49) with (9) we see that indeed the action integral is not invariant under this subgroup, so that a Noether-type conserved quantity does not exist. Nevertheless, the equation of motion is invariant under the subgroup, since the Lagrangian is simply multiplied by a constant. It therefore follows that solutions are transformed into solutions under this subgroup.

The transformation properties of the Lagrangian under the groups generated by G_7 and G_8 are somewhat more complicated. For the generator G_7 we have, from (42):

$$\xi = q \sin t, \tag{50a}$$

$$\eta = q^2 \cos t. \tag{50b}$$

From (36) we find that, for the Lagrangian (19):

$$E\{L\} = (2q \cos t)L - (2\dot{q} \sin t)(\dot{q}^2 + q^2) \tag{51}$$

and from (37)

$$L' = (3q \cos t)L - (\dot{q} \sin t)(\dot{q}^2 + 3q^2). \tag{52}$$

We thus have, for the first-order change in L, from (32*a*):

$$\tilde{L} = L - \theta[(3q \cos t)L - (\dot{q} \sin t)(\dot{q}^2 + 3q^2)]. \tag{53}$$

Because of our method of derivation, it is clear that the Euler equation associated with (52) must lead to the harmonic oscillator equation of motion. Carrying out the calculation, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L'}{\partial \dot{q}}\right) - \frac{\partial L'}{\partial q} = (\ddot{q} + q)3C_4 \tag{54}$$

where C_4 is the conserved quantity defined by (26*d*). Setting the right-hand side of (54) to zero, we obtain

$$\ddot{q} + q = 0 \tag{55a}$$

and

$$C_4 = 0. \tag{55b}$$

Equation (55*a*) is precisely the harmonic oscillator equation of motion; furthermore, the general solution of (55*b*) is

$$q = A \sin t,$$

which also satisfies (55*a*). Therefore no spurious solutions are introduced by the second factor in (54), and the Lagrangian L' leads to the same solutions as the Lagrangian (19).

The situation is similar for the generator G_8; in this case

$$L' = -(3q \sin t)L - (\dot{q} \cos t)(\dot{q}^2 + 3q^2).$$

It is well known that any Lagrangian of the form

$$L = (\dot{q}^2 - q^2)K + \dot{f}(q, t), \tag{56}$$

where K is a constant, leads to the same Euler equation as the harmonic oscillator Lagrangian (19), and is therefore said to be equivalent to it. As we have seen, however, there exist Lagrangians which are equivalent to (19) without being of the form (56). The existence of harmonic oscillator Lagrangians not of the form (56) has previously been noted by Rosen (1969), who has given a procedure for producing a certain class of them. The Lagrangians discussed by Rosen are functions only of q and $\dot{q}$, and are therefore distinct from those we have described, which depend explicitly on the time.

Finally, it is noteworthy that the operators

$$J_1 = G_4 - G_8, \tag{57a}$$

$$J_2 = G_3 + G_7, \tag{57b}$$

$$J_3 = G_5, \tag{57c}$$

form the Lie algebra of the three-dimensional rotation group. This subgroup of the harmonic oscillator symmetry group has been used (Wulfman and Wybourne 1976) to investigate the periodicity of solutions, without using the properties of the solutions themselves. It is interesting that one must use G_7 and G_8 to form the algebra (57); thus, if attention had been confined only to those groups which yield conserved quantities via Noether's theorem, the subgroup (57) would not have been available. This seems to indicate that for certain considerations it may be advantageous to deal with the complete symmetry group of a system, rather than just the Noether subgroup.

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